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# ASYMPTOTIC MOTIONS AND STABILITY OF THE ELASTOPLASTIC OSCILLATOR STUDIED VIA MAPS

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**Abstract**—The response of the hysteretic oscillator subjected to periodic actions presents aspects that are not yet completely clarified. In this paper the elastoplastic oscillator is studied with tools that are typical of nonlinear dynamics as developed in recent years. Using iterated maps, including the Poincaré map, it is shown how the dynamics concerned can be reduced to a one-dimensional problem. An extended numerical survey shows how the response of the elastoplastic oscillator is always asymptotically stable, with the exception of a zero measure set in the space of control variables where it is only stable. Moreover solutions having a period other than that of the external force do not exist, though for some parameter values higher harmonics may assume a very relevant role.

## INTRODUCTION

Much effort has been devoted to attempting to understand nonlinear one-dof oscillator behaviour under harmonic forcing, generally assuming very simple constitutive laws, often polynomials of second or third order [see e.g. Guckenheimer and Holmes (1983) and Wiggins (1990)]. Hysteretic oscillators have received much less attention because their treatment is not easily framed in the context of existing analytical techniques. Hysteretic constitutive laws are not represented by single-valued functions in displacement, and also, in many of the fields where nonlinear dynamics has taken root and is better developed, purely hysteretic behaviour is not common.

Hysteretic behaviour is very important in civil engineering. However, periodic forces are not so common and there are not very many studies of the periodic response of the hysteretic oscillator in this field. Nowadays, new challenges and new knowledge make the study of the response of hysteretic oscillators to harmonic forces worth studying more closely; considering also that comprehension of behaviour under sinusoidal input may lead to better understanding of behaviour under different excitations. Among hysteretic oscillators the elastoplastic oscillator, on account of its simplicity and its historical importance, has a special role. The earliest studies used mainly approximate analytical techniques such as the harmonic balance or slowly varying parameters (Caughey, 1960; Jennings, 1964; Iwan, 1965). In a more recent paper by Masri (1975) a simple technique was developed that gives exact results for the most common loading–unloading cycle situations, in which no intermediate unloading occurs. Eventually, with the introduction of an iterated map in Miller and Butler (1988) a satisfactory numerical technique was developed to evaluate the stationary response and some progress was also made regarding stability. The intermediate unloading was shown to occur for only a limited portion of the state parameter space. Though many aspects of the elastoplastic response to harmonic forces are known—namely that the response has an unbounded resonance above a critical value of the force, and that in some situations higher harmonics may be relevant—some questions concerning stability and the various possible kinds of periodic motions still remain unanswered.

In all the studies mentioned the constitutive relation is modelled in terms of force and displacement. Because of the non-single-valuedness of this function, concepts such as phase space, Poincaré map, Floquet theory, etc. cannot be applied directly. In this paper the

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elastoplastic oscillator is modelled in an incremental form which leads to a phase space with force and velocity as variables [see e.g. Capecchi (1991)]. In this way the problem is reduced to the study of a system of two equations defined by ordinary functions, as is commonly the case in classical nonlinear dynamics; many useful concepts can then be applied directly. The first part of this paper deals with theoretical aspects; the appropriate phase space is introduced and it is shown that the displacement is a dependent variable and that the oscillator dynamics are essentially one-dimensional. Iterated maps are then defined including the Poincaré map and some one-dimensional maps. The strategy developed in Miller and Butler (1988) is given a sounder foundation and new formulations are found that allow the periodic response and its stability to be studied with very little computational effort. Applications are given in the last section. The extended numerical survey enables an answer to be given to the still unclarified phenomena concerning the stability and the possibility of subharmonic orbits.

## EQUATION OF MOTION

The motion of an oscillator having a constitutive law with memory, such as the elastoplastic one, cannot be described by an ordinary second order differential equation. A more general equation is required, defined by functional operators that can, in principle, be written as:

$$\ddot{x}+f = p(t), \tag{1}$$

where $x$, $f$ and $p$ are respectively the response, restoring force and excitation, and proper nondimensionalization is chosen. The symbol $f$ stands for a functional of $x(t)$ instead of a function.

This approach does not allow the use of the state phase concept; nor, consequently, can the problem be studied in the classical framework of nonlinear dynamics. For this reason it is useful to restate eqn (1) more suitably. If only the elastoplastic oscillator (EPL) is considered, the motion can still be represented by an ordinary differential equation, but of third order, where the state variables are not only $x$ and $\dot{x}$ but also $\ddot{x}$. Indeed, $f$ is in any case a function of $t$ with derivative $\dot{f}$ that can be given as a function of $f$ and $\dot{x}$ in the form:

$$\dot{f} = h(\dot{x}, f) = \begin{cases} 0 & f \operatorname{sign}(\dot{x}) \geq 1, \\ \dot{x} & f \operatorname{sign}(\dot{x}) < 1. \end{cases} \tag{2}$$

The function $h(\dot{x}, f)$ defined above is shown in Fig. 1. Though having a high degree of

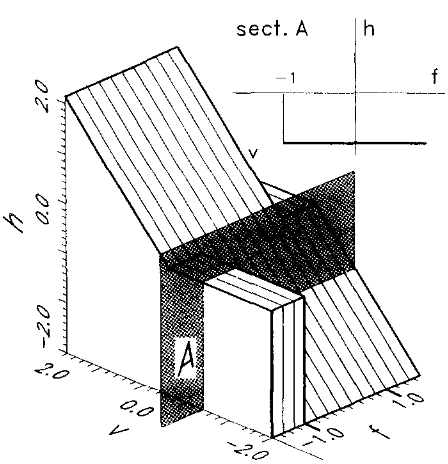


Fig. 1. Surface $h(v, f)$ for EPL.

irregularity, it is a proper function of $\dot{x}$ and $f$, i.e. it is single valued. The differential equation of third order is obtained from the derivative of eqn (1) with respect to time:

$$\dddot{x}+h(\dot{x},p-\ddot{x}) = \dot{p}(t), \tag{3}$$

where the expression for $f$ as provided by (1) is made explicit in $h(\dot{x}, f)$. Equation (3) defines the motion together with the initial conditions $x(t_0)$, $\dot{x}(t_0)$, $\ddot{x}(t_0)$ at a given time $t_0$. If $x$ is not considered an independent variable, eqn (3) can also be seen as a second order differential equation in the velocity $v = \dot{x}$, that needs the two initial conditions $v(t_0)$ and $\dot{v}(t_0)$. In the following $x$ will always be considered as a dependent variable and the elastoplastic oscillator will be studied in a two-dimensional phase space. The variable $x$, where needed, is obtained through a numerical integration of $\dot{x}$.

In nonlinear dynamical systems the problem of motion is usually framed within the context of vector fields. Here it is assumed that time $t$ is not a phase variable so that the vector field contains $t$ as a parameter. To obtain the vector field for EPL, eqn (3) must be rewritten as a system of two first order differential equations. The most natural choice would be to assume $a = \ddot{x}$ and $v$ as phase variables, however reference to $f$ and $v$ is more convenient, because in this way the nature of the phase space as a manifold $M$ with boundaries is self-evident. With $\dot{x} = v$ eqns (1) and (2) read:

$$\begin{aligned} \dot{v} &= p(t)-f, \\ \dot{f} &= h(v,f). \end{aligned} \tag{4}$$

The time dependent vector field is then defined by $X_t$: $(t, v, f) \rightarrow [(v, f), (p(t)-f, h(v,f))]$, where $v$ and $f$ take values in $M = \{(v, f)|v \in R, f \in [-1, 1]\}$. The problem thus formulated presents strong analogies with the impact problem where the boundaries of the phase space manifold are given by stops, see e.g. Foale and Bishop (1992). The elastoplastic system may indeed be considered as a double sided impacting system having a vanishing coefficient of restitution and a delayed release.

Equations (4) show that the EPL dynamics can be represented in the standard form in the same way as a classical nonlinear elastic oscillator. Its main peculiarities, apart from the fact that $x$ is no longer a state variable, both lie in the nature of the phase space $M$ and in the vector field $X_t$. Manifold $M$ is characterized by the presence of a boundary, while $X_t$, because of the low smoothness of $h(v, f)$ is only $C^{-1}$. These two circumstances do not allow the automatic application of many fundamental theorems of nonlinear dynamics. An immediate effect of $X_t$ being only $C^{-1}$ is that there is no one-to-one relation between a point $p \in M$ and its image $q \in M$ due to the flow $\phi_{t,\tau}$ of the vector field $X_t$ because $h(v, f)$ violates the Lipschitz condition with respect to $f$. Notice that the use of the word *flow* is slightly improper because $\phi_{t,\tau}$ is not a diffeomorphism on $M$; here it simply means that $\phi_{t,\tau}$ is the map that moves points $p$ at instant $t$ to points $q$ at instants $t+\tau$. The absence of a one-to-one correspondence between $p$ and $q$ can be heuristically deduced from the examination of $\dot{f} = h(v, f)$, which is an ordinary differential equation in $f$ when $v(t)$ is given. Figure 2 shows some possible paths emanating from various $p_i$, with increasing initial values for $f$, to reach the single final position $q$.

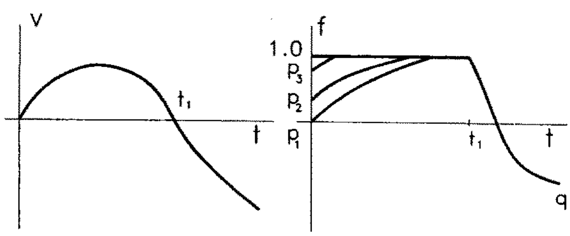


Fig. 2. Time-histories of $f(t)$ for assigned $v(t)$.

## ITERATED MAPS

For the case of harmonic functions $p(t)$, the vector field $X_t$ reads:

$$\dot{v} = \gamma \cos \omega t - f,$$
$$\dot{f} = h(v, f), \tag{5}$$

where the coefficient $\gamma$ stands for the dimensionless intensity of the forcing action and $\omega$ is its frequency, normalized to the elastic frequency of the oscillator. The main aspects of oscillator dynamics can be evidenced without an explicit study of the vector field $X_t$ if the problem is formulated using a map defined in $M$. A typical map is the Poincaré map, $P_{t_0}$:

$$P_{t_0}: M \to M,$$
$$v(t_0), f(t_0) \to v(t_0+T), f(t_0+T),$$

where $t_0$ is an arbitrary time and $T = 2\pi/\omega$. $P_{t_0}$ is referred to as the Poincaré map based at $t_0$. It only depends on the two control parameters $\gamma$ and $\omega$. Its iterated application to a point $p$ is equivalent to stroboscopically sampling the point's trajectory at an interval $T$. An orbit of a point $p \in M$ is given by the set $O(p) = \{P^i_{t_0}(p) | i \in Z\}$. When $O(p)$ is finite, $p$ is called periodic and the smallest integer $n$, so that $p = P^n_{t_0}(p)$, is called the period of $p$ and $O(p)$ a subharmonic orbit of period $n$. Order $n$ periodic points are obtained from fixed points of $P^n_{t_0}$, namely by solving an algebraic (not a differential) equation:

$$P^n_{t_0}(p) - p = 0. \tag{6}$$

For a classical nonlinear oscillator, $P_{t_0}$ is a diffeomorphism of $M$ of the same class as the vector field; moreover, the choice of $t_0$ is trivial because two maps $P_{t_0}$ and $P_{t_1}$ are conjugate, namely there is a diffeomorphism $g$ in $M$ such that $g \circ P_{t_0} = P_{t_1} \circ g$. This means that the eigenvalues of the tangent map $T_p P_{t_0}$ do not depend on $t_0$. For the elastoplastic oscillator $P_{t_0}$ is no longer a diffeomorphism. In particular $P_{t_0}$ is continuous and endowed with a tangent map $TP_{t_0}$ which is also continuous, with the exception of a few points. However it does not admit an inverse because it does not define a one-to-one correspondence. The conjugacy property is instead maintained.

The domain of $P_{t_0}$ is the noncompact manifold $M$. If only the asymptotic behavior is of interest, then the study can be restricted to a compact set $M^*$. This can be seen better by referring to the vector field. Whatever the initial conditions are, an instant $t^*$ exists when the oscillator moves out of a plastic phase with $v = 0$ and $f = \pm 1$; from now on $v$ cannot exceed a maximum value $\tilde{v}_{t^*}$. The global maximum $\tilde{v}$ of $\tilde{v}_{t^*}$ is bounded; it is obtained by evaluating $\tilde{v}_{t^*}$ by integrating (5) with initial conditions $v(t^*) = 0, f(t^*) = \pm 1$, and varying $t^*$ in $[0, T]$ (which gives all the possibilities due to the periodicity of the excitation).

The non-invertibility of $P_{t_0}$ at each point $p$ of $M$ implies that $T_p P_{t_0}$ is singular having rank equal to one, and thus a zero eigenvalue. The image $S = P_{t_0}(M)$ is not a submanifold of $M$. It is however a collection of a few one-dimensional submanifolds whose tangent space at a point $P_{t_0}(p)$ is the eigenvector of $T_p P_{t_0}$ associated with the non-zero eigenvalue. Figure

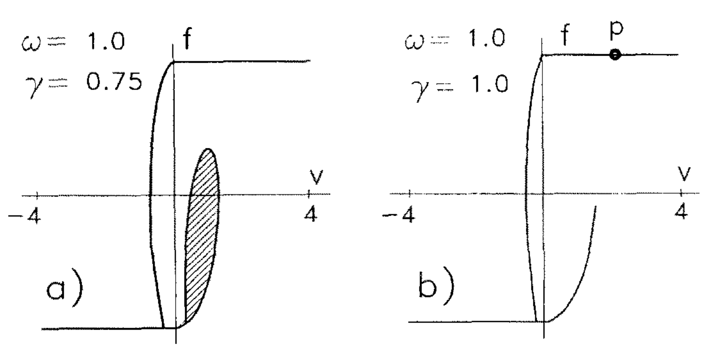


Fig. 3. Image of the phase plane under $P_{t_0}$.

3 shows images $S^*$ of $M^*$ under $P_{t_0}$ for $t_0 = 0.273$, $\omega = 1.0$ and $\gamma = (0.75, 1.0)$. In the figure sharp zones are evident indicating discontinuity of $TP_{t_0}$, moreover there are points where $S^*$ is not locally homeomorphic to $R$ because more than one branch departs from them. The curve $S$ is the locus of points obtained by integrating (5) over a duration $T-\tau$, starting from initial conditions $(0, \pm 1)$, where $\tau$ is the time interval needed to leave the plastic state first. More precisely, $S$ is the union of two curves $C^+$ and $C^-$. Curve $C^+$ is composed of the straight line $(v, 1)$ and a curve that is represented parametrically by $(v, f) = \phi_{t_0+\tau, T-\tau}(0, 1)$ with $\tau \in [0, T]$; $C^-$ is identical but with opposite sign for the force. Assuming as an initial condition $(0, -1)$, a point $p(\tau) \in C^-$ immediately escapes from the plastic phase, otherwise $p(\tau) \in C^+$. For low values of $\gamma$ the image of $M$ also includes those open sets of points that represent initial conditions that do not become plastic during a period $T$. They are indicated with a shaded region in Fig. 3(a).

The prevailing one-dimensional nature of EPL dynamics suggests that it is possible to study the main characteristic of the motion with maps defined over one-dimensional sets (or possibly manifolds). If the time $t_0$ is chosen in such a way that the fixed point is inside a plastic region, like point $p$ of Fig. 3(b), a neighbourhood having a nonzero measure exists for which it is possible to define a map by referring to the state variable $v$ only, because integration over a period always returns $f$ to the limit value $\pm 1$. The search for periodic orbits of EPL is reduced to the solution of the fixed point problem:

$$Q_{t_0}(v) - v = 0. \tag{7}$$

$Q_{t_0}$ is none other than the Poincaré map with contracted domain and range. The practical application of eqn (7) has some drawbacks because the subinterval of $[0, T]$ for correct values of the base time $t_0$ has to be estimated *a priori*.

A more general one-dimensional map is obtained by referring to the image $S$ of $M$. Because all the periodic orbits of $P_{t_0}$ belong to $S$, the whole dynamics of EPL could in principle be studied with a map over this set. Consider a point $p_0 \equiv 3 \in C^-$ [Fig. 4(a)], defined by the parameter value $\tau_0$ of its parametric representation $(v, f) = \phi_{t_0+\tau, T-\tau}(0, -1)$ (Point $p_0$ is obtained starting from 0 at $t_0$, entering the plastic phase in 1 and leaving it at $t_0 + \tau$ from 2). The image $p_1 \equiv 6 = P_{t_0}(p_0)$ corresponds to the instant $\tau_1$, path 3-4-5. This may be evaluated from the time $\Delta\tau_p$ needed to pass from $(0, -1)$, assumed as initial condition at instant $\tau_0$, to the new exit from the plastic phase $(0, -1)$, path 2-3-4-5: $\tau_1 = \Delta\tau_p - (T - \tau_0)$. Similar considerations are valid for curve $C^+$. A relation between $\tau_0$ and $\tau_1$ is thus established. If $\tau_1$ and $\tau_0$ are considered apart from the modulus $T$, it is possible to define a one-dimensional map: $G: [0, T] \to [0, T]$ that may possess fixed points. These fixed points correspond to periodic solutions with an *a priori* unknown period, that must be evaluated by applying $P_{t_0+\tau_0}$ to the point $(0, \pm 1)$. Loosely, $G$ is a map that transforms a given instant $\bar{t}_0 = t_0 + \tau_0$ corresponding to the initial condition $(0, \pm 1)$ to an instant $\bar{t}_1 = t_0 + \tau_1$ for which the couple $(0, \pm 1)$ again appears. With this meaning $G$ appears similar to the map reported in Miller and Butler (1988). Notice that $G$ is discontinuous in $[0, T]$

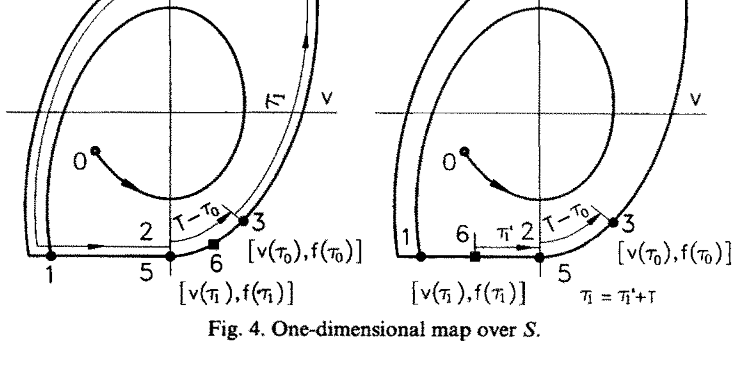


Fig. 4. One-dimensional map over $S$.

because $C = C^+ \cup C^-$ and the passage from $C^+$ to $C^-$ means assuming a discontinuity of $\tau$: the same point $p$ has two different values for the parameter $\tau$ according to whether it is considered in $C^+$ or in $C^-$. This reparametrization can be avoided as follows: let $p_0(\tau_0) \in C^-$ be projected by $P_{t_0}$ into $C^+$. However $P^n_{t_0}(p_0) \in C^-$ for some $n > 0$, because the fixed points certainly belong to $C^+ \cap C^-$. The relation between the first passage $\tau'_1$ through $(0, -1)$ and $\tau_0$ and $\Delta\tau_p$ is the same as before, except for the modulus $T$: $\tau'_1 = \Delta\tau_p - (nT - \tau_0)$. Of course the map $G$ remains discontinuous even with this slight change. The situation is made clear in Fig. 4, where points $p_1 \equiv 6$ which are close to 5, in cases (a) and (b), but have very different parameter values $\tau_1$.

A different point of view suggests choosing the kind of fixed point that is wanted, namely the couple of values $v$, $f$ and searching for the value of $t_0$ for which $P_{t_0}(v, f) = (v, f)$. Values of $v$ and $f$ certainly reached during an elastoplastic cycle are $(0, -1)$, or $(0, +1)$, corresponding to the exit from a plastic state. The dynamics of $P_{t_0}$ is then reduced to the search for values of $t_0$ for which $P^n_{t_0}(0, \pm 1) = (0, \pm 1)$. The algorithmic implementation of this concept, which results in two equations and one unknown, is a minimum problem with the objective function:

$$l(t_0) = \| P^n_{t_0}(0, \pm 1) - (0, \pm 1) \| \qquad (8)$$

where the Euclidean norm is assumed. From a technical point of view it may however be convenient to seek the solution of a single equation in $t_0$:

$$Q^n_{t_0}(0) = 0 \qquad (9)$$

and to verify that $f = \pm 1$.

The use of the map $G$ could be helpful when a primarily analytical approach is pursued, because the simple nature of the constitutive law for EPL allows a simple expression of $t_1 = G(t_0)$, though not in an explicit form. For numerical purposes it would lead to a more complex problem than that given by eqns (7), (8) and (9) because the definition of $G$ requires the computation of the unloading point, which requires the solution of a nonlinear equation. Moreover $G$ is not a continuous function over $[0, T]$, even though it is generally continuous around a fixed point. If the previous concepts are to be extended to many dof systems the use of the Poincaré map with a known fixed point and unknown base is preferable on account of its higher numerical efficiency and its greater conceptual simplicity.

## MAP EVALUATION

The various maps introduced must be integrated over the vector field. In principle this can be accomplished with any suitable method such as the fourth order Runge–Kutta method. Numerical experience has shown that for the elastoplastic oscillator an explicit integration technique, such as the Runge–Kutta, gives satisfactory results only if the integration is accomplished using very small steps. This is because of the singularity of $h(v, f)$. For some parameter values $\gamma$ and $\omega$ where the response is very complex, a step of $1000/T$ must be adopted, making numerical analysis very expensive. In order to eliminate any possible source of uncontrolled errors and to save computation time, an exact integration technique is used here.

The oscillator response is divided into elastic and plastic phases. In each phase the equation of motion is linear; using $v$ as variable, it is obtained by deriving the first of equations (5):

$$\begin{aligned} \ddot{v} + v &= -\gamma\omega \sin \omega t \quad \text{ELASTIC}, \\ \ddot{v} &= -\gamma\omega \sin \omega t \quad \text{PLASTIC}. \end{aligned} \qquad (10)$$

The difficulty of integration is shifted to the evaluation of instants $t_i$ and values $v_i$ where a change of state occurs. The conceptual separation of plastic and elastic phases not only enables the implementation of an efficient numerical tool, but also furnishes theoretical arguments. If $L_{t_i,\tau_i}$ and $P_{t_i,\tau_i}$ indicate the linear elastic and plastic operators that provide $(v,f)$ after the interval $\tau_i$, starting from $(v,f)_{t_i}$, the Poincaré map is the product of operators:

$$P_{t_0} = \cdots \circ L_{t_i,\tau_i} \circ P_{t_{i+1},\tau_{i+1}} \circ L_{t_{i+2},\tau_{i+2}} \circ \cdots \tag{11}$$

This is a chain whose structure depends on the actual motion of EPL. Because the operators that appear are continuous in $(v,f)$ and in $t_i$, $P_{t_0}$ is continuous. For the same reason $TP_{t_0}$ is also continuous, except in those points that are critical for the chain structure (11). A point $p$ is singular for $TP_{t_0}$ if $p+\Delta p$ corresponds to a different operator chain.

Where symmetric solutions are concerned, only reduced maps need be considered: $P_{t_0}^{1/2}$, $R^{1/2}$, so that $P_{t_0}^{1/2} \circ P_{t_0}^{1/2} = P_{t_0}$, $G^{1/2} \circ G^{1/2} = G$, requiring half the computational effort. Similar considerations are also valid for eqns (8) and (9). Map $G^{1/2}$ deserves a comment. This is the map that brings time from instant $t_0$, corresponding to $p_- = (0,-1)$, to instant $t_1$ corresponding to $p_+ = (0,1)$. Note that though points $p_+$ and $p_-$ generally alternate regularly, sequences $p_+p_+p_-p_-$ are possible, as reported in Miller and Butler (1988).

## STABILITY ANALYSIS

The stability of a fixed point $p$ of a map $W$ is generally checked by looking at the eigenvalues $\lambda_i$ of $T_pW$. If $|\lambda_i| < 1\ \forall\, i$, then $p$ is asymptotically stable; if, on the contrary, $\exists\, i$ s.t. $|\lambda_i| > 1$, then $p$ is unstable. This appears evident if the linear approximation of $W$, around $p$ is considered:

$$W_{t_0}(p+\xi) = p + T_pW\xi + O(\xi^2). \tag{12}$$

These considerations apply to EPL as well, where the definition of $W$ varies according to whether $p$ is a point within $M$ or a boundary point. In the first situation $W = P_{t_0}$ is two-dimensional and a stability check calls for the numerical evaluation of $T_pP_{t_0}$, computing $\Delta P_{t_0}/\Delta v$, $\Delta P_{t_0}/\Delta f$, for $\|\Delta v\|$, $\|\Delta f\| < \delta_\varepsilon$, and then evaluating eigenvalues $\lambda_i$, one of which is zero. In the second situation $W = Q_{t_0}$ is one-dimensional. The single eigenvalue is evaluated with the relation $\lambda = [Q_{t_0}(v+\Delta v) - v]/\Delta v$, for $\|\Delta v\| < \delta_\varepsilon$. If the fixed points are evaluated as discussed in previous sections neither the first situation nor the second can be applied, because $t_0$ is a transition instant and $T_pW$ is singular. In practice it is enough to rescale $W$ by choosing an instant $t'_0$ where the fixed point has $v \neq 0$, preferably on the boundary $\partial M$, where stability analysis is easier. Because of the conjugacy of the Poincaré maps based at different instants $t_0$, it follows that the nonzero eigenvalue of $T_pW_{t_0}$ is the same as that of $T_pW_{t'_0}$, so that there is no ambiguity in the use of different maps.

Stability can also be checked by referring to map $G$. It first must be shown that a fixed point $p = (0,-1)$ is stable for $P_{t_0}$ if, and only if, the fixed point $\tau_0$ of $G$ is stable. Let $p_0 = (0,-1)$ be the fixed point whose stability is under study, and $U_p$ a neighbour with diameter $\delta_p$. As $P_{\tau_0}$ is continuous for each $p \in U_p$, $P_{\tau_0}^{1/2}$ gives a point $q \in U_q$, where $U_q$ is a neighbour of $q_0 = (0,1)$ of diameter $\delta_q$ that can be arbitrarily reduced by reducing $\delta_p$. It is then possible to find $\varepsilon$ so that $\phi_{T/2,\varepsilon} = q_0$, and there is a one-to-one correspondence $q = g(\varepsilon)$ that is continuous and well defined for $\varepsilon > 0$ and $\varepsilon < 0$, because all points belong to the forward integrated orbit. With reference to Fig. 5 the Poincaré map $P_{\tau_0}$ can be expressed as:

$$P_{\tau_0} = \phi_{\tau_0,T/2} \circ \phi_{\tau_0+T/2,\varepsilon} \circ \phi_{\tau_0+T/2+\varepsilon,t_1} \circ \phi_{\tau_0+T-\varepsilon',\varepsilon'}(p), \tag{13}$$

where $\phi_{\tau_1,\tau_2}$ is the phase flux from $\tau_1$ with duration $\tau_2$ and $t_1 = G^{1/2}(\tau_0+\varepsilon) - \tau_0$. The first term moves $p$ to $q$, which is delayed by $\varepsilon$ with respect to $q_0$. The second term leads from $q$ to $q_0$, the third leads to $p_0$, and the last term eventually completes the map ($\varepsilon' = T/2 - t_1 - \varepsilon$), leading to $p'$. If $\tau_0$ is a stable fixed point for $G$, then $G^{1/2}$ is a contractive map and $|\varepsilon'| < |\varepsilon|$.

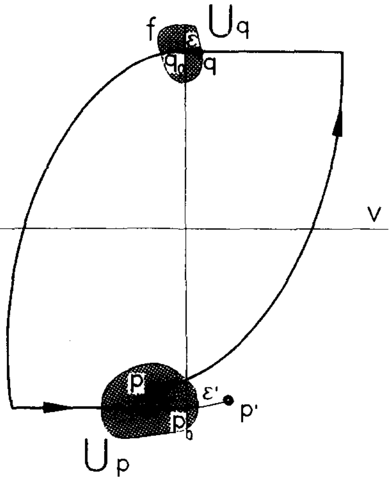


Fig. 5. Relation between stability of $G$ and $P_{t_0}$.

For the continuity of $g(\varepsilon)$, point $p'$ resulting from application of $\phi_{\tau_0+T-\varepsilon',\varepsilon'}$ is linked to $q$ by $\|p'-p_0\| < \|q-q_0\|$, as $|\varepsilon'| < |\varepsilon|$. Map $P_{\tau_0}^{1/2}$ is thus contractive as, consequently, is $P_{\tau_0}$. Similarly, if $\tau_0$ is unstable so is $p_0$.

## APPLICATIONS

Before beginning to study the stationary response and stability of EPL, it will be shown how various formulations of the iterated map interpret the same phenomenon. In Fig. 6 trends of $G$ and function $l(t_0)$ are reported for $\gamma = 1.05$ and various values of $\omega$. For greater clarity reference is made to a situation where higher harmonics are relevant. Other maps are similar in structure though more regular. Map $G$ is discontinuous in $[0, T]$ and, for $\omega = 0.35$, admits a fixed point $t_0^{(a)}$ with an eigenvalue $\lambda = 1$, so that it is at the boundary of stability. For $\omega < 0.35$ the fixed point of $G$ abruptly moves to the right, though it remains stable; the same occurs for $\omega > 0.35$. There is no generic bifurcation behaviour because

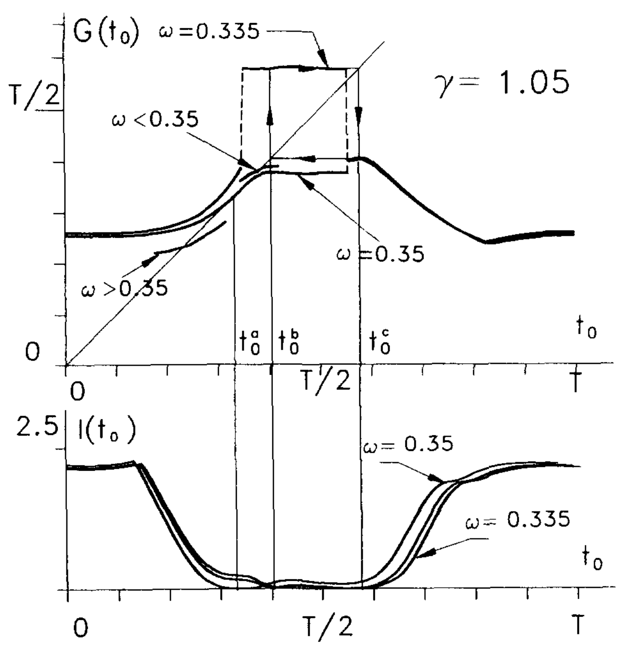


Fig. 6. Fixed points of $G$ and zeroes of $l(t_0)$.

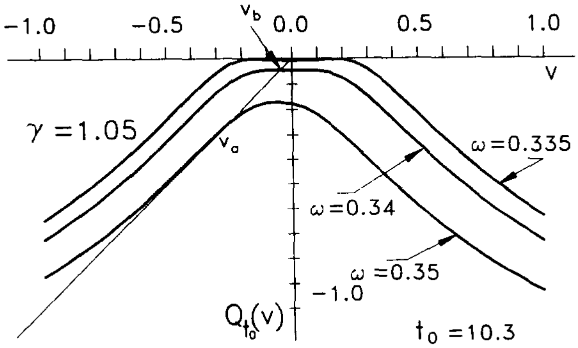

Fig. 7. Fixed points of $Q_{t_0}$.

$\partial G/(\partial\omega, \partial t_0)$ does not meet the necessary transversality condition (Wiggins, 1990). For $\omega = 0.335$ the map $G$ no longer admits fixed points. A scrutiny of the figure shows that two fixed points $t_0^{(b)}$ and $t_0^{(c)}$ exist for $G^2$. This means that the unique periodic response of EPL corresponding to these two fixed points possesses four loading and unloading elastic–plastic phases. A numerical survey shows that, in fact, only fixed points of $G$ and $G^2$ exist and the latter are limited to a very thin portion of the control variable space, in agreement with Miller and Butler (1988). The trend of $l(t_0)$ confirms the previous results and considerations. Values of $t_0$ for which $(0, -1)$ is a fixed point of $P_{t_0}$ coincide exactly with fixed points of $G, G^2$.

Figure 7 reports the trend of $Q_{t_0}$ for $t_0 = 10.30$ and the values of $\omega$ and $\gamma$ previously considered. $Q_{t_0}$ also shows an eigenvalue $\lambda = 1$ for $\omega = 0.35$; now the map has a fixed point for $\omega = 0.335$ as well. Here the eigenvalue is very small as evidenced by the flat shape of $Q_{t_0}$ around the fixed point $v^b$. This circumstance indicates how efficiently the fixed points of $Q_{t_0}$ could be evaluated with the Newton algorithm. A single iteration is sometimes enough for a very accurate solution. The regular trend of $Q_{t_0}$ is maintained, too, for $\omega$ close to the critical value $\omega = 0.35$.

## FREQUENCY RESPONSE CURVES

The most common representation of the periodic response of a one-dof oscillator is the frequency-response curve (f.r.c.) for various values of the excitation amplitude $\gamma$. In principle f.r.c.s may possess various branches. One defines the main branch as that which goes towards the stationary elastic curve for low amplitude. Other branches are called bifurcated or isola according to whether they are connected to the main branch or disjointed.

Step-by-step integrations show how the harmonic response of the oscillator assumption, made in earlier approximate studies (Caughey, 1960; Capecchi and Vestroni, 1985, 1990; DebChaudhury, 1985) is largely unsatisfactory in some situations. Figure 8 illustrates time-histories of velocity and force and orbits in the phase space $v, f$ for various values of $\gamma$ and $\omega$, from which it is clear that the harmonic motion assumption is not always admissible. The same figure also shows that the other assumption often made, concerning the modalities of loading and unloading—namely no intermediate unloadings from the plastic state—is incorrect. It is thus clear that an accurate numerical analysis such as that developed here is required.

The main branch of exact f.r.c.s for the EPL oscillator is already described in the literature referring to the variable $x$ (Miller and Butler, 1988; Capecchi, 1991). Because of the central role played here by the variable $v$ the f.r.c.s are included for the sake of completeness in Fig. 9 with reference to this last variable. For low values of $\gamma$ exact and approximate curves differ only slightly. For $\gamma = 1$ the behaviour near the superharmonic regions becomes complex and resonances for $\omega = 1/3, \omega = 1/5, \ldots$ are evidenced. For $\gamma > 1$ unbounded resonances occur for $\omega = 0$, in contrast to the harmonic balance method, which predicts unbounded resonance for $\gamma > 4/\pi$. A further increase of $\gamma$ makes the superharmonic resonances gradually disappear until, for $\gamma > 1.35$, the frequency response curve

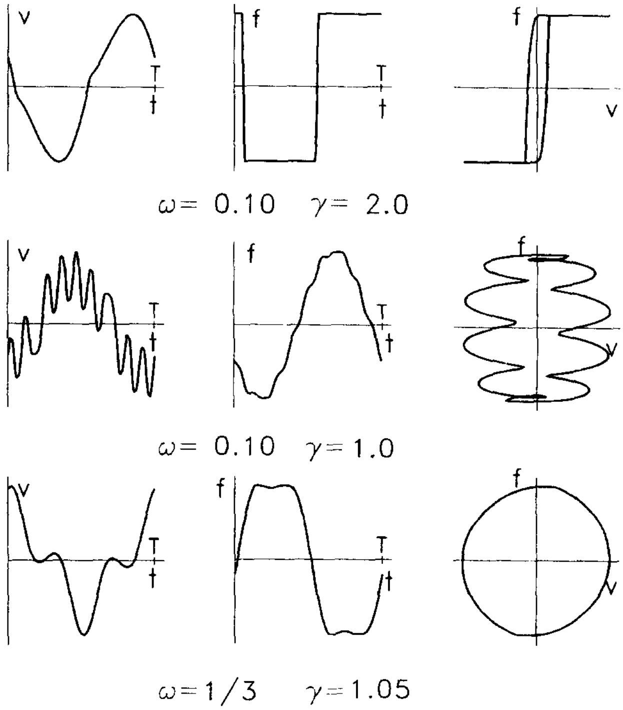


Fig. 8. Phase plots for various $\gamma$ and $\omega$ values.

appears as a hyperbole having the two coordinate axes as asymptotes. The harmonic balance method achieves the same result for $\gamma > 4/\pi$.

## OTHER SOLUTIONS AND STABILITY

In oscillators with a two-dimensional phase space the numerical search for all the $n$-T periodic stable and unstable solutions, which in principle are infinite, cannot generally be pursued by looking for all the fixed points of the Poincaré map $P^n_{t_0}$, because the algorithms needed to solve the algebraic equations (6) always require *a priori* estimates of the solution, which are not usually precise enough. For this reason a cell-to-cell mapping procedure is preferred [see e.g. Hsu (1987)], which can partition the phase space into attracting basins, i.e. the sets of initial conditions that, under iterated mapping, lead to the various stable

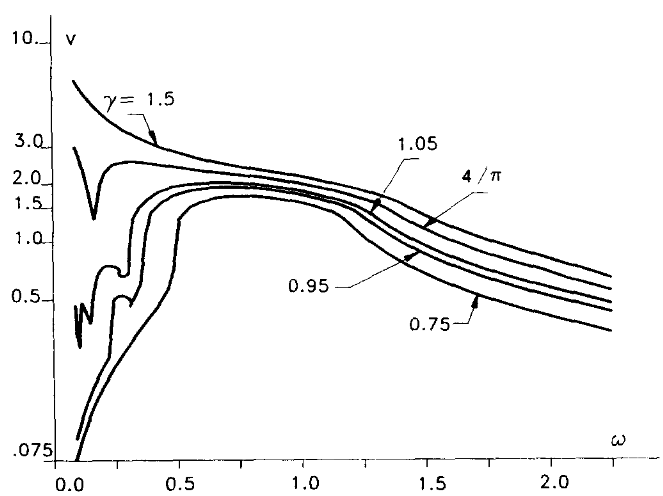


Fig. 9. Frequency-response curves in terms of velocity.

periodic solutions. The systematic exploration of the control parameters $\omega$ and $\gamma$ using this procedure, though feasible, is very time consuming.

For the elastoplastic oscillator, the treatment adopted circumvents the need to consider initial conditions in the phase space since, as shown, all the possible periodic solutions are furnished by the solutions of eqn (9), for $t_0 \in [0, T]$. Although these solutions are achieved using the Newton method, which requires an initial estimate of $t_0$, the difficulties are greatly reduced by the one-dimensional nature of the problem, which can be tackled efficiently. It is then possible to explore an exhaustive set of values for the control variables $\omega$ and $\gamma$, so that the presence of stable or unstable solutions other than those belonging to the main branch (with the possibility of different periods) and the stability of the periodic solutions, as yet unclarified in the literature, can be verified with a high degree of reliability, though not certainty.

In practice, to find the solutions of $Q^n_{t_0}(0) = 0$, the interval $[0, T]$ of variation of $t_0$ is partitioned into small parts, a step $\Delta t_0 = T/250$ being considered sufficient. For each value of $t_0$ the vector field (5) is integrated exactly over $n$ periods and the value of $Q^n_{t_0}(0)$ is checked. When a zero is approximately localized, its position is defined more precisely with the Newton method. Subharmonics of period $n = 1, 2, 3, 4, 5$ were sought by varying $\gamma, \omega$ in a fairly large domain of the control parameter space. More precisely for $\omega \in [0, 2.5]$, $\gamma$ is varied in $[0, 5]$ with a grid $\Delta\omega = \Delta\gamma = 0.025$. For $\omega \in (2.5, 10]$, $\gamma$ is varied according to the law $\gamma = (\omega^2 - 1) * \beta$, with $\beta \in [1, 5]$ and a grid $\Delta\omega = 0.2$ and $\Delta\beta = 0.1$; this is because the oscillator response for lower values of $\gamma$ would be elastic. No fixed points with a period greater than $T$ were found and each solution was verified as being symmetric and belonging to the main branch. Of course the numerical survey can not answer the theoretical questions regarding the existence of other fixed points, because of the possible fractal structure of the parameter space, which is not evidenced numerically. However, the study can be considered exhaustive from a technical point of view.

The stability was also checked, simply by evaluating the single eigenvalue $\lambda$ of $Q_{t_0}$. Figure 10 reports the trend of $\lambda$ versus $\omega$ for two values of $\gamma$. It can be seen that $\lambda$ is always lower than 1. The only exceptions are when the response is elastic (where it is known that the solution is stable, though not asymptotically because of zero damping) and in some isolated points where stability cannot be assured on the basis of the eigenvalues alone. For these, a numerical simulation, carried out by integrating over 500 periods starting from slightly different initial conditions from the fixed points under examination, confirms that the solution is stable. The fact that all points on the main branch are stable means that there can be no bifurcated paths of non-symmetric 1-T periodic solutions, which could generate a sequence of subharmonic orbits with period $2^s T$, for $s = 1, 2, 3, \ldots$. This agrees with the numerical observation that no subharmonic motion exists.

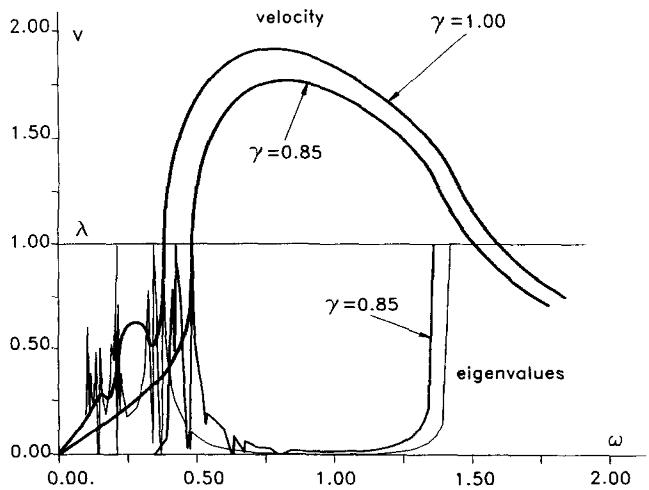


Fig. 10. Eigenvalues of $Q_{t_0}$ versus $\omega$.

## CONCLUSIONS

Though the restoring force of an elastoplastic oscillator is not a single-valued function in the displacement $x$, by adopting an incremental formulation —$\dot{f}$ instead of $f$—it is possible to describe its motion by means of ordinary differential equations, in the space of state variables $v$ and $f$. This formulation allows the use of concepts and tools typical of nonlinear classical mechanics, such as the Poincaré map or more general iterated maps. It greatly simplifies the problem of motion because it is essentially one-dimensional in nature, on account of the peculiar characteristics of the elastoplastic oscillator, and it is possible to use one-dimensional maps, which save computations in the evaluation of periodic motion and stability.

Because of the efficiency of the algorithm an extended numerical survey allowing clarification of many aspects is feasible. The motion is always asymptotically stable, with the exception of a few points where it is only stable. All possible orbits have the same period as the force, so no subharmonic motion is possible. For $\gamma$ close to unity, pronounced superharmonic effects are evident.

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